Spherical and planar three-dimensional anti-de Sitter black holes

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Abstract.

The technique of dimensional reduction was used in a recent paper [1] where a three-dimensional (3D) Einstein-Maxwell-Dilaton theory was built from the usual four-dimensional (4D) Einstein-Maxwell-Hilbert action for general relativity. Starting from a class of 4D toroidal black holes in asymptotically anti-de Sitter (AdS) spacetimes several 3D black holes were obtained and studied in such a context. In the present work we choose a particular case of the 3D action which presents Maxwell field, dilaton field and an extra scalar field, besides gravity field and a negative cosmological constant, and obtain new 3D static black hole solutions whose horizons may have spherical or planar topology. We show that there is a 3D static spherically symmetric solution analogous to the 4D Reissner-Nordström-AdS black hole, and obtain other new 3D black holes with planar topology. From the static spherical solutions, new rotating 3D black holes are also obtained and analyzed in some detail.

PACS numbers: 04.70.-s, 04.40.-b, 04.20.Jb.

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1. Introduction

The early works in three-dimensional (3D) general relativity showed that it provides a test-bed to 4D and higher-D theories [2, 3, 4, 5, 6]. Moreover, the work on 3D gravity theories has seen a great impulse after the discovery that 3D general relativity possesses a black hole solution, the BTZ black hole [7, 8]. This black hole is a solution of the Einstein-Hilbert action including a negative cosmological constant term Λ. One can also show that the BTZ black hole can be constructed by identifying certain points of the 3D anti-de Sitter (AdS) spacetime [8, 9].

The BTZ solution arouse the interest on the subject of 3D black holes and a whole set of new solutions in 3D followed from a number of different dilaton-gauge vector theories coupled to gravity. For instance, upon reducing 4D Einstein-Maxwell theory with Λ and with one spatial Killing vector it was shown in [10, 11] that it gives rise to a 3D Brans-Dicke-Maxwell theory with its own black hole, which when reinterpreted back in 4D is a black hole with a toroidal horizon. One can then naturally extend the whole set to Brans-Dicke theories [12, 13]. Other solutions with different couplings have also been found [14, 15] (see [13] for a more complete list).

A renewal of interest in 3D black hole solutions in asymptotically AdS spacetimes came after the AdS-conformal field theory conjecture (AdS/CFT) [16]. ADS/CFT tells us how to compute at strong coupling in quantum field theories using the near-horizon geometry of certain black branes. The 3D black hole solutions we investigate in the present paper can be important in the verification of the conjecture. It is known that the BTZ black hole play an important role in this context, since many higher-D extreme black holes of string theory have a near-horizon geometry containing the BTZ black hole [17]. Other 3D black holes may also be important in connection to higher-D string theory such that they can be interpreted as the near horizon structures of a brane rotating in extra dimensions [18]. The solutions we study here are generalizations of the 3D black holes found in reference [1] including rotation, and have the BTZ black hole as a particular case.

The motivation to study lower dimensional black holes is also provided by the evidence that some physical properties of black holes such as Hawking radiation and evaporation can be more easily understood in the context of two and three dimensions [19, 20]. As we shall see, the 3D black holes we study below have geometric properties very close to some 4D solutions such as the Reissner-Nordström black hole. The study of these 3D black holes may then help us to better understand the ordinary black hole physics.

On the other hand, dilaton gravity theories are of particular interest since they emerge as low-energy limit of string theories. They are commonly studied in lower dimensions because the lower-dimensional Lagrangian yields field equations whose solutions are easily found and analyzed.

The usual starting point to find solution in a given dimension, D say, is the action of gravity theory with the generic dilaton and gauge vector couplings in that dimension.

One then derives the corresponding equations of motion, tries an ansatz for the solution and then finally from the differential equations finds the black hole solutions compatible with the ansatz.

From the known solutions in that dimension one can find new solutions in the same dimension if the theory possesses dualities symmetries that convert one solution into another in a nontrivial way [21].

New solutions in lower dimensions can be obtained through dimensional reduction, where one can reduce a theory by several dimensions. Among the procedures to perform a dimensional reduction, the classical Kaluza-Klein reduction [22], and the Lagrangian dimensional reduction [23] are commonly used, and are both equivalent when reducing through one symmetric compact direction (see e. g. [21, 22, 23, 24, 25, 26]).

In reference [1] the classical Kaluza-Klein procedure was used to find new black hole solutions in 3D. A 3D Lagrangian including gravity, cosmological constant, dilaton field, Maxwell gauge field, Kaluza-Klein gauge field, and an extra scalar field was obtained from the 4D Einstein-Maxwell gravity theory. Such a 3D Lagrangian is a Brans-Dicke-Dilaton-Maxwell gravity theory with the dilaton playing a special role. In particular, the dimensional reduction through the Killing azimuthal direction $\partial/\partial\varphi$, of rotating charged black holes with toroidal topology was considered. The resulting 3D black hole displays an isotropic (i.e., circularly symmetric) horizon, with two gauge charges and a number of other interesting properties.

On could also expect to obtain a second class of three-dimensional spherical and even anisotropic (non-spherical) black hole solutions through the dimensional reduction of 4D spacetimes such as the Kerr-Newman-AdS family of black holes. The reduction can be performed through the Killing azimuthal direction $\partial/\partial\varphi$ and, because of the special topology of the Kerr-Newman-AdS black hole, an analytical continuation through the polar angle θ has to be made. However, this procedure conduces to ill-defined solutions that do not have a natural interpretation from a three-dimensional point of view. These kind of problems can be avoided by following different procedures as done for instance in [27], where a 3D distorted black hole was obtained through dimensional reduction of the Schwarzschild black hole.

Other way of avoiding the above mentioned problems of topology is starting from the 3D dilaton-gravity action, obtained through the dimensional reduction from 4D Einstein-gravity, and working entirely in 3D dimensions. This is the strategy followed in the present work, where a particular case of the 3D dilaton-action written in [1] is used to build different classes of solutions.

In the next section it is presented a summary of the model, the action and the field equations are written, and the definitions of global charges are reviewed.

Assuming that the spacetime admits one timelike Killing vector and one spacelike Killing vector, the metric is then static and has a spatial symmetry that can be spherical or planar depending on the topology of the spacetime. We make an ansatz for the other fields that are allowed to be non-spherical. This is done in section 3, by using Schwarzschild-like coordinates (t, r, θ), and assuming that the metric depends only on

the radial coordinate r, while the other fields depend also on the coordinate θ. It is then showed that there are solutions to the field equations representing spacetimes endowed with spherical black holes and with non-spherical dilaton field.

Five different classes of solutions presenting black holes are reported in the following sections. We show that the solution presented in section 4 is in fact a particular case of the 3D black hole studied in [1]. The other cases are new 3D black hole solutions. A planar black hole is obtained in section 5 by assuming the coordinate θ to be non-periodic. The spherically (circularly) symmetric spacetime analyzed in section 6 is similar to the 4D Reissner-Nordström-AdS black hole. Considering a different topology of the subspace with constant values of r and t, the same solution analyzed in section 6 may be interpreted as a planar black hole. This is briefly discussed in section 7. Other planar black hole solution is presented in section 8. The rotating versions of the new 3D spherical black holes are considered in section 9. In section 10 we conclude.

2. 3D Dilaton-Maxwell gravity

2.1. The 3D Lagrangian and field equations

The Kaluza-Klein dimensional reduction of 4D black holes was used in [1] to build new 3D spacetimes with black holes. In this section we summarize the results obtained there for static black holes and review the interpretation issue of the reduced solutions as 3D black holes.

It is assumed that the 4D manifold $\mathcal{M}_4$ can be decomposed as $\mathcal{M}_4 = \mathcal{M}\times S^1$, with $\mathcal{M}$, S^1, being the 3D manifold and the circle, respectively. For the sake of simplicity we particularize the discussion to the case of static 4D spacetimes.

The four-dimensional Lagrangian $\hat{\mathcal{L}}$ is assumed to be the usual Einstein-Hilbert-Maxwell Lagrangian with cosmological term $\hat\Lambda$, and electromagnetic field $\hat{\mathbf{F}} = \mathrm{d}\hat{\mathbf{A}}$ (where $\hat{\mathbf{A}}$ is the gauge field), given by (we use geometric units where $G = 1$, $c = 1$).

$$\hat S = \int dx^4\, \hat{\mathcal{L}} = \frac{1}{16\pi}\int d^4x\sqrt{-\hat g}\left(\hat R - 2\hat\Lambda - \hat{\mathbf{F}}^2\right). \qquad (1)$$

The convention adopted here is that quantities wearing hats are defined in 4D and quantities without hats belong to 3D manifolds.

In order to proceed with the reduction of the Lagrangian $\hat{\mathcal{L}}$ to 3D, consider a 4D spacetime metric admitting one spacelike Killing vector, ∂_φ, where φ is assumed to be a compact direction. In such a case the 4D static metric may be decomposed in the form

$$d\hat s^2 = \Phi^{2\beta_0} ds^2 + \Phi^{2\beta_1} d\varphi^2\,,$$

where ds^2 is the 3D metric, Φ and all the other metric coefficients are functions independent of φ, and β_0 and β_1 are real numbers.

We are free to choose β_0 and β_1, i.e., we are free to choose the frame in which to work, with the different frames being related by conformal transformations. As discussed in detail in reference [1], the most interesting frame is the one that fixes $\beta_0 = 0$ with β_1

free. In this paper we choose $\beta_1 = 1$. The dilaton field Φ^2 is assumed to be nonzero, except at isolated points where it may be zero.

It is well known that different frames, related by conformal transformations, are physically inequivalent, e.g., one frame can give spacetime singularities where the other does not (see, e.g., [28, 29, 30]). Therefore, once a choice of parameters β_0 and β_1 has been made, then a particular frame has been chosen, and the metric in this frame is interpreted as describing the physical spacetime. Other choices of parameters lead to solutions whose spacetime may have different physical and geometric properties.

To dimensionally reduce the electromagnetic gauge field we do

$$\hat{\mathbf{A}} = \mathbf{A} + \Psi \mathbf{d}\varphi \,,$$

where for compact φ the gauge group is $U(1)$. In the last equation $\mathbf{A}$ is a 1-form while Ψ is a 0-form. From this we can define the 3D Maxwell field

$$\mathbf{F} = \mathbf{dA} \,.$$

The Kaluza-Klein dimensional reduction procedure then gives

$$S = \frac{L_3}{16\pi} \int \mathcal{L}\, d^3x = \frac{L_3}{16\pi} \int d^3x \sqrt{-g\Phi^2} \left[R + 6\alpha^2 - \mathbf{F}^2 - 2\Phi^{-2} (\nabla\Psi)^2 \right], \quad (2)$$

where L_3 is the result of integration along the φ direction. The 3D cosmological constant is defined as $\Lambda = \hat{\Lambda}$, and we have written $\alpha^2 \equiv -\Lambda/3$.

The equations of motion which follow from the action (2) for the graviton $\mathbf{g}$, the gauge field $\mathbf{A}$, the dilaton Φ, and the scalar Ψ are, respectively,

$$\begin{aligned} G_{ij} = {} & 3\alpha^2 g_{ij} + \Phi^{-1} \left(\nabla_i \nabla_j \Phi - g_{ij} \nabla^2 \Phi \right) + 2 \left(F_{ik} F_j{}^k - \frac{1}{4} g_{ij} \mathbf{F}^2 \right) \\ & + 2\Phi^{-2} \left(\nabla_i \Psi \nabla_j \Psi - \frac{1}{2} g_{ij} (\nabla\Psi)^2 \right), \end{aligned} \quad (3)$$

$$\nabla_j \left(\Phi\, F^{ij} \right) = 0 \,, \quad (4)$$

$$\nabla_i \left(\nabla^i \Phi \right) = +3\alpha^2\, \Phi + \frac{1}{2} \Phi\, F^2 - \Phi^{-1} \left(\nabla\Psi \right)^2 \,, \quad (5)$$

$$\nabla_i \left[\Phi^{-1} \nabla^i \Psi \right] = 0 \,, \quad (6)$$

where G_{ij} is the Einstein tensor.

2.2. The global charges

For the sake of convenience, we write here the definitions of gravitational mass, angular momentum, and charges in the 3D spacetime arise by using the formalism of Brown and York [31] modified to include a dilaton and other fields [32, 33, 34]. The conventions adopted are the same as in [1].

(i) Conventions: We assume that the 3D spacetime $\mathcal{M}$ is topologically the product of a spacelike surface D_2 and a real line time interval I, $\mathcal{M} = D_2 \times I$. D_2 has the topology of a disk. Its boundary ∂D_2 has the topology of a circle and is denoted by $\mathcal{S}_1$. The boundary of $\mathcal{M}$, $\partial\mathcal{M}$, consists of two spacelike surfaces $t = t_1$ and $t = t_2$, and a timelike surface

$\mathcal{S}_1 \times I$ joining them. Let t^i be a timelike unit vector ($t_i t^i = -1$) normal to a spacelike surface D_2 (that foliates $\mathcal{M}$), and n^i be the outward unit vector normal to the boundary $\partial\mathcal{M}$ ($n_i n^i = 1$). Let us denote the spacetime metric on $\mathcal{M}$ by g_{ij} ($i,j = 0,1,2$). Hence $h_{ij} = g_{ij} + t_i t_j$ is the induced metric on D_2 and $\sigma_{ij} = g_{ij} + t_i t_j - n_i n_j$ is the induced metric on $\mathcal{S}_1$. The induced metric on the spacetime boundary $\partial\mathcal{M}$ is $\gamma_{ij} = g_{ij} - n_i n_j = \sigma_{ij} - t_i t_j$. We also assume that the spacetime admits the two Killing vectors needed in order to define mass and angular momentum: a timelike Killing vector $\eta^i_t = (\frac{\partial}{\partial t})^i$ and a spacelike (axial) Killing vector $\eta^i_\theta = (\partial_\theta)^i$.

(ii) Mass: By adapting the Brown and York procedure to take into account the dilaton field the following definition of mass M on a 3D spacetime admitting a timelike Killing vector η_t is obtained

$$M = -L_3 \int_{\mathcal{S}_1} \delta\left(\epsilon^\Phi\right) t_i \eta^i_t \, d\mathcal{S} \,, \tag{7}$$

where $d\mathcal{S} = \sqrt{\sigma}\, d\theta$ with θ being a coordinate on $\mathcal{S}_1$, and σ being the determinant of the induced metric on $\mathcal{S}_1$. ϵ^Φ is energy surface density on $\mathcal{S}_1$. Recall that $\epsilon^\Phi = \frac{\kappa^\Phi}{8\pi}$, where κ^Φ is the trace of the extrinsic curvature of $\mathcal{S}_1$ as embedded on D_2, modified by the presence of the dilaton (see below).

(iii) Angular momentum: Similarly to the mass, the definition of angular momentum J for a 3D spacetime admitting a spacelike Killing vector $\partial_\theta = \eta_\theta$ can also be modified to include the dilaton. For the cases we are going to consider here, a sufficiently general definition is

$$J = L_3 \int_{\mathcal{S}_1} \delta\left({j^\Phi}_i\right) \eta^i_\theta \, d\mathcal{S} \,, \tag{8}$$

where ${j^\Phi}_i$ is the momentum surface density on $\mathcal{S}_1$, modified by the presence of the dilaton. The angular momentum density may be written as $j^\Phi_i = -2\sigma_{ij} n_l {P_\Phi}^{jl}/\sqrt{h}$, where P^{jl}_Φ is the canonical gravitational momentum, modified by the presence of the dilaton [33]. The other quantities were defined above, and are viewed as tensor quantities in the two-space D_2.

To define κ^Φ and j^Φ_i explicitly we consider the case when the 3D metric can be split as

$$\begin{aligned} ds^2 &= -\, N^2 dt^2 + h_{mn}\left(dx^m + N^m dt\right)\left(dx^n + N^n dt\right) \\ &= -\left(N^2 - R^2 (N^\theta)^2\right) dt^2 + 2R^2 N^\theta d\theta\, dt + f^{-2} dr^2 + R^2 \left(d\theta + V\, dr\right)^2 \,, \end{aligned} \tag{9}$$

where $m,n = 1,2$, $x^1 = r$, $x^2 = \theta$, and θ parameterizes $\mathcal{S}_1$. Functions N, N^θ, f, R and V depend on coordinate r only. κ^Φ may then be written as

$$\kappa^\Phi = -\frac{1}{2} f \left[2\frac{\partial\Phi}{\partial r} + \Phi\left(\frac{2}{R}\frac{\partial R}{\partial r} - \nabla_\theta V\right)\right] \,, \tag{10}$$

where $\nabla_\theta V$ is the covariant derivative on $\mathcal{S}_1$, which in the present case is identically zero.

Now, the angular momentum density is given by

$$j_i^{\Phi}\eta_\theta^i = \frac{1}{8\pi}\frac{\partial N^\theta}{\partial r}\frac{R^2}{N}\sqrt{f\Phi^2\left(1+R^2V^2 f\right)}\,. \tag{11}$$

(iv) Electric charge of the gauge field A_i: The gauge charge can be obtained by the Gauss law, adapted to non-asymptotically flat stationary spacetimes [31], and to the presence of the dilaton (see also [10, 11])

$$Q_e = \frac{L_3}{4\pi}\int_{\mathcal{S}_1}\delta\left(E_i\right)n^i d\mathcal{S}\,, \tag{12}$$

where $E_i \equiv \Phi\, F_{ij}t^j$.

The magnetic charge is not defined here since the 3D spacetimes we consider below do not contain magnetic Dirac-like monopoles.

(v) Dilatonic charges: Stationary asymptotically AdS 3D black holes may also be characterized by the dilaton charge. In fact, two different definitions can be formulated [35, 36]. However, dilaton charges are of little importance in our analysis because, for the particular dilatonic black holes considered here, they are identically zero.

(vi) Charges of the scalar field Ψ: In this case, the equation of motion for Ψ yields two conserved currents. It is then possible to define the corresponding conserved charges (of electric and magnetic type, respectively) as

$$Q_\Psi = \frac{L_3}{4\pi}\int_{\mathcal{S}_1}\delta\left(\frac{1}{\Phi}\nabla^i\Psi\right)\,n_i\,d\mathcal{S}, \tag{13}$$

$$\tilde{Q}_\Psi = \frac{L_3}{4\pi}\int_{\mathcal{S}_1}\epsilon_{ijk}\delta\left(\nabla^k\Psi\right)\,t^i n^j d\mathcal{S}\,. \tag{14}$$

It is straightforward to verify that both of the above definitions represent conserved charges. From equation (6) it is possible to identify the quantity $J^i_\Psi = \frac{1}{\Phi}\nabla^i\Psi$ as a conserved current, $\nabla_i J^i_\Psi = 0$. The corresponding conserved charge (analog to the electric charge) being (13). The second charge (14) is also conserved for it follows from the conserved current $\nabla^j\left(\epsilon_{ijk}\nabla^k\Psi\right)$.

3. 3D static black hole solutions

The dimensional reduction of 4D Kerr-Newman-AdS black-holes suggests we can find new 3D spherical (and non-spherical) black hole solutions. Assuming the spacetime is static, i. e., there is a timelike Killing vector, we can choose Schwarzschild-like coordinates (t, r, θ) in which the metric can be written in the diagonal form as showed in the Appendix, whose coefficients depend on coordinates r and θ only.

If we restrict the metric coefficients to depend only upon one of the radial coordinate r, i.e., the 3D spacetime admits also a spacelike Killing vector, the resulting metric can be written as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2 d\theta^2\,, \tag{15}$$

where $F(r)$ is an arbitrary function.

Coordinates t and r are defined in the ranges: $-\infty < t < +\infty$, and $0 \leq r < +\infty$. Therefore, the above metric contains two different topologies of interest in the present work, depending on the topology of C_θ the hyper-surfaces (lines) with constant values of t and r. If θ is a compact direction (θ is an angular coordinate), C_θ is the circle S^1, and (15) represents a spherically symmetric spacetime. On the other hand, if θ is a non-compact direction ($-\infty < \theta < +\infty$), C_θ is the real line $\mathbf{R}$ and the spacetime is said to have planar symmetry.

The other fields are allowed to depend on both of the spacelike coordinates. Namely, $A_i = A_i(r,\theta)$, $\Phi = \Phi(r,\theta)$ and $\Psi = \Psi(r,\theta)$. As shown explicitly in the Appendix, the field equations admit solutions where the functional dependence of the fields upon θ is nontrivial, even though the severe restrictions imposed by the symmetry of the metric. Such restrictions follow directly from the Einstein equations (3). The functional dependence of the fields on r and θ has to be such that the total energy momentum tensor (EMT) results independent on θ. That is to say, the right hand side of equation (3) must depend on the radial coordinate only, even though it also involves functions and derivatives with respect to the angular coordinate. These compatibility conditions will be automatically satisfied by any given solution of the field equations.

The task of finding solutions is more easily accomplished by separating variables through the ansatz

$$\mathbf{A}(r,\theta) = A_1(r)\, A_2(\theta)\mathbf{d}t\,, \tag{16}$$

$$\Phi(r,\theta) = \Phi_1(r)\Phi_2(\theta)\,, \tag{17}$$

$$\Psi(r,\theta) = \Psi_1(r)\Psi_2(\theta)\,, \tag{18}$$

where the new functions $A_1(r)$, $A_2(\theta)$, $\Phi_1(r)$, $\Phi_2(\theta)$, etc., depend only on one variable, as indicated. This choice for the gauge field $\mathbf{A}$ excludes 3D spacetimes with magnetic charge such as the cases considered in Refs. [37, 38, 39], but it is sufficiently general for the purposes of the present analysis.

The field equations (3)–(6) then give seven equations for the unknowns F, A_1, A_2, Φ_1, Φ_2, Ψ_1 and Ψ_2. The relevant equations can be cast into the form presented in the system (86)–(93) in the Appendix. Some of the equations can be easily integrated and we are left with the following set of equations

$$\frac{F'}{2}\left(\frac{1}{r}+\frac{\Phi_1'}{\Phi_1}\right) - 3\alpha^2 + {A_1'}^2 + \frac{F}{r}\frac{\Phi_1'}{\Phi_1} + \frac{\pm k^2}{r^2} + \frac{1}{r^2}\frac{g^2}{\Phi_1^2} = 0\,, \tag{19}$$

$$A_1' = \frac{q_0}{\Phi_1\, r}\,, \tag{20}$$

$$A_2 = \text{constant}\,, \tag{21}$$

$$\Phi_1 = p_0 r\,, \tag{22}$$

$$\ddot{\Phi}_2 = \pm k^2 \Phi_2\,, \tag{23}$$

$$\Psi_1 = \text{constant}\,, \tag{24}$$

$$\dot{\Psi}_2 = g\Phi_2\,, \tag{25}$$

where k^2, q_0, p_0 and g are integration constants, and the prime stands for the derivative with respect to r.

Function $\Phi_1(r) = p_0 r$ gives the dependence of the dilaton upon the coordinate r. According to equation (17), the arbitrary constant p_0 can be made equal to unity

As discussed in the Appendix, the constant $\Psi_1(r)$ can also be made equal to unity without loss of generality, since it is absorbed by $\Psi_2(\theta)$. Moreover, equations (20) and (21) tell that the gauge field $\mathbf{A}$ is a function of r only. Also, the constant A_2 can be normalized to unity.

The above equations furnish the complete solutions for all the unknown fields. Φ and Ψ are given by (23) and (25), respectively. Equations (20) and (21) then furnish A_1 which specifies the electromagnetic potential in its entirety (it is spherically symmetric). With these functions at hand we then finally solve equation (19) and check the second Einstein equation, (87) of the Appendix, for compatibility. The result is that the above ansatz admits five different classes of solutions characterized by a particular dilaton field and by a particular topology, which we enumerate and nominate as follows:

(i) *Spherically symmetric dilaton and spherical 3D black holes*: $\Phi(r,\theta) = c_0\, r$;

(ii) *Non-spherically symmetric dilaton and planar 3D black holes-I*: $\Phi(r,\theta) = r(c_0\theta + c_1)$;

(iii) *Circular dilaton function - 3D Reissner-Nordström-AdS black holes*: $\Phi(r,\theta) = r(c_0 \sin(k\theta) + c_1 \cos(k\theta))$;

(iv) *Circular dilaton function and planar 3D black holes-II*: $\Phi(r,\theta) = r(c_0 \sin(k\theta) + c_1 \cos(k\theta))$;

(v) *Hyperbolic dilaton function and planar black holes-III*: $\Phi(r,\theta) = r(c_0 \sinh(k\theta) + c_1 \cosh(k\theta))$;

c_0 and c_1 being arbitrary integration constants which assume different values in each one of the three different cases. In the cases (i) and (iii), θ is a periodic coordinate ($\theta \in [0, 2\pi]$), while in (ii), (iv) and (v), it is nonperiodic ($-\infty < \theta < +\infty$).

The names spherical, non-spherical, circular and hyperbolic are used in reference to the functional form of the dilaton field $\Phi(r,\theta)$. The name Reissner-Nordström-AdS is given in reference to those well known black holes our new solutions can be related to. Other interesting naming is suggested by the topology of the corresponding 4D black holes, as explained in the following sections, where we comment on each one of these cases.

As we have already mentioned, the case Φ = constant gives the BTZ black hole solution (see the Appendix). This black hole has been extensively studied in the literature and we do not consider it here.

For future reference we write here the contributions to the EMT from the fields $\mathbf{A}$, Φ and Ψ

$$(T)_t^{\,t} = T_r^{\,r} = {A_1'}^2 - \frac{{\Phi_1}'}{\Phi_1}\left(\frac{F}{r} + \frac{F'}{2}\right) - \frac{1}{r^2}\frac{\ddot{\Phi}_2}{\Phi_2} - \frac{1}{r^2}\frac{\dot{\Psi}_2^2}{{\Phi_2}^2}\,, \tag{26}$$

$$(T)^r_\theta = (T_A)^\theta_r\, r^2 F = \frac{1}{r^2}\frac{\dot{\Phi}_2}{\Phi_2}\left(\frac{\Phi_1{}'}{\Phi_1} - \frac{1}{r}\right), \tag{27}$$

$$(T)^\theta_\theta = -A_1{}'^2 - \frac{1}{r^2}\frac{\dot{\Psi}_2^2}{\Phi_2^2} + \frac{\Phi_1{}'}{\Phi_1}F', \tag{28}$$

the other components of the EMT being zero. The dependence of the EMT on the angular coordinate is through the second derivative of the dilaton filed with respect to that variable only. From equations (23) and (25), it follows that the EMT is a continuous function of θ. Moreover, using the result from equation (22) it is seen that $T_r^\theta = 0$, and all the other non-zero components of the energy-momentum tensor are continuous functions of the r coordinate only, as expected. Also, the Lagrangian dependence on θ is carried completely by the dilaton field $\Phi_2(\theta)$.

Now we investigate in some detail the various different solutions which follow from the above model.

4. Spherical black hole solutions

This case is $\Phi = r$. The result for the other scalar field is $\Psi =$ constant, which is obtained from equation (99) in the Appendix by choosing $c_0 = c_1 = 0$. It should be noted that considering the angular coordinate as a normal azimuthal angle, the general scalar fields given by equations (99) are not periodic, and thereby not single-valued functions. The 3D dilaton field corresponds to a metric field in the 4D spacetime, which has to be a nonzero well defined function of the coordinates. Hence, in order to avoid possible problems due to the multivalued character of the dilaton field Φ we have to take $c_0 = 0$. In fact, the condition $\Phi_2(\theta) = \Phi_2(\theta + 2\pi)$ implies $c_0 = 0$.

The scalar field Ψ corresponds to a gauge potential in 4D spacetime and, as it is well known from gauge field theory, classically it does not have any direct physical meaning. However, in 3D spacetime, once we have fixed the dilaton field to be spherical, the only gauge freedom left to the scalar potential Ψ is an arbitrary additive constant. Therefore, by imposing the periodicity condition $\Psi(\theta) = \Psi(\theta + 2\pi)$ it follows $g = 0$. The result is $\Psi =$ const., which can be made equal to zero.

The metric and nonzero fields in this case are

$$ds^2 = -\left(\alpha^2 r^2 - \frac{2m}{r} - \frac{q^2}{r^2}\right)dt^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{2m}{r} + \frac{q^2}{r^2}} + r^2 d\theta^2, \tag{29}$$

$$\mathbf{A} = -\frac{q}{r}\mathbf{d}t, \tag{30}$$

$$\Phi = r. \tag{31}$$

Once the coordinates are defined in the intervals $-\infty < t < +\infty$, $0 \le r < +\infty$, and $0 \le \theta \le 2\pi$, the present solution is a particular case of the 3D charged black hole studied in [1], obtained through dimensional reduction of the 4D toroidal rotating black hole. The properties of this black hole follow by putting $a = 0$ and $g = 0$ in the 3D black hole studied there. The resulting spacetime has spherical (circular) symmetry.

For the sake of completeness, and for future comparison, we summarize here the main properties of such a spacetime.

It is straightforward to show that m, is the mass, q is the electric gauge charge, source to the field $\mathbf{A}$. The other charges and the angular momentum are zero.

The Ricci and Kretschmann curvature scalars are, respectively,

$$R = -6\alpha^2 - \frac{2q^2}{r^4}\,, \tag{32}$$

$$K = 12\alpha^4 + \frac{8\alpha^2 q^2}{r^4} + \frac{24m^2}{r^6} - \frac{64mq^2}{r^7} + \frac{44q^4}{r^8}\,, \tag{33}$$

showing that there is a singularity at $r = 0$.

The horizons of (29) are given by the real roots of the equation

$$\alpha^2 r^4 - 2mr + q^2 = 0\,. \tag{34}$$

In the analysis of horizons, the relevant function is $Descr = 27m^4 - 16q^6\alpha^2$ and we have three different cases: (i) If $Descr > 0$ equation (34) has two real positive roots corresponding to two horizons, the event horizon at r_+ and the Cauchy or inner horizon at r_-. The singularity at $r = 0$ is enclosed by both horizons. The spacetime can then be extended through the horizons till the singularity. It represents a 3D static black hole. (ii) $Descr = 0$ – the solution is the extreme black hole spacetime. There is only one horizon at $r = r_+ = r_- = \sqrt[3]{m/2\alpha^2} = \sqrt[4]{\frac{4}{3}\frac{q^2}{\alpha^2}}$. The singularity is hidden (to external observers) by the horizon. Geodesic inward lines end at the singularity $r = 0$. (iii) $Descr < 0$ – this solution has no horizons and represents a naked singularity.

5. Planar 3D black hole solutions - I

There is a second possible interpretation for the solution presented in equation (99) of the Appendix, besides the spherical black hole of the preceding section. Namely, the case where the spatial sections (t = constant) of the spacetime have the topology of a plane.

If we assume the ranges of the spatial coordinates as $0 \le r < \infty$, and $-\infty < \theta < +\infty$, then the spacetime is a "planar" 3D black hole. I. e., the topology of C_θ, the spacetime section defined by t = constant and r = constant, is $\mathbf{R}$ (the real line), which in the 2D spatial section of the spacetime looks like a one-dimensional wall.

In such a case, it is useful defining a new coordinate $z = L\theta$, with L being a positive constant carrying dimensions of length (an interesting choice is to take $L = 1/\alpha$ as in Ref. [11]). The scalar fields are then of the form

$$\Phi = r(c_1 z + c_o)\,, \tag{35}$$

$$\Psi = g\left(c_1 z^2/2 + c_o z\right) + c_2\,; \tag{36}$$

and the relevant metric coefficient is

$$F = \alpha^2 r^2 - \frac{2m}{r} - \frac{q^2 + g^2}{r^2}\,. \tag{37}$$

Both of the scalar fields Φ and Ψ, as well as the metric field F, are well behaved functions of the coordinates (except at the singularity $r = 0$).

As far as the energy momentum tensor is concerned, there are no discontinuities like strings or domain walls in the spacetime. Moreover, the physical quantities associated to the fields $\mathbf{A}$, Φ and Ψ, are also well defined everywhere (except at the singularity). The total mass and charges are infinite, but the parameters m, q, etc., are finite (per unit length) quantities. To see that explicitly let us calculate, for instance, the mass of the black hole. The boundary S_1 [see equation (7)] in the present case coincides with the one-dimensional region C_θ in the limit $r =\longrightarrow \infty$, and is not compact. Therefore, in order to avoid infinite quantities during calculations we select just a finite region of the boundary between $z = z_1$ and $z = z_2 > z_1$, and put $L = z_2 - z_1$. Equation (7) then gives

$$M_L = \frac{L_3}{4\pi} m \int_0^L c_1 z \, \frac{dz}{L} = mL \,,$$

where we have chosen $L_3 = 8\pi/c_1$. The total mass is infinite since $L \longrightarrow \infty$. However the mass per unit length $M_L/L = m$, which appears in the metric, is a well defined finite quantity. Similarly, the other charges per unit length of the black hole can be determined (see also the next section).

The curvature invariants are the same as for the spherical black hole considered in the previous section, given in equations (32) and (33), with q^2 replaced by $q^2 + g^2$. Namely,

$$R = -6\alpha^2 - \frac{2(q^2+g^2)}{r^4}\,, \tag{38}$$

$$K = 12\alpha^4 + \frac{8\alpha^2(q^2+g^2)}{r^4} + \frac{24m^2}{r^6} - \frac{64m(q^2+g^2)}{r^7} + \frac{44\,(q^2+g^2)^2}{r^8}\,. \tag{39}$$

The singularity at $r = 0$ is also a one-dimensional wall. Because of the singularity, the spacetime cannot be extended to the region $r < 0$.

The horizons of the planar spacetime are given by the real roots of the function F given by equation (37): $\alpha^2 r^4 - 2mr + q^2 + g^2 = 0$. This solution can then be called a planar black hole (or a black wall) spacetime. Here, the relevant function is $Descr = 27m^4 - 16(q^2 + g^2)^3\alpha^2$ and we have three different cases, as in section 4: (i) Black hole with event and Cauchy horizons ($Descr > 0$), (ii) Extreme black hole ($Descr = 0$), (iii) and naked singularity ($Descr < 0$).

6. 3D Reissner-Nordström-AdS black hole solutions

The solution in the case we choose $\Phi = r\,(c_0 \sin k\theta + c_1 \cos k\theta)$ is

$$ds^2 = -\left(k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\right) dt^2 + \frac{dr^2}{k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}} + r^2 d\theta^2\,, \tag{40}$$

$$\mathbf{A} = -\frac{q}{r}\mathbf{d}t\,, \tag{41}$$

$$\Phi = r\,(c_0 \sin k\theta + c_1 \cos k\theta)\,, \tag{42}$$

$$\Psi \;\; = \frac{g}{k}\left(-c_0\cos k\theta + c_1 \sin k\theta\right) + c_2 \tag{43}$$

If θ is interpreted as the azimuthal angle, then the topology of C_θ (the subspace with $t =$ const., $r =$ const.) is the circle S^1, and the solution is a 3D analogous to the 4D Reissner-Nordström-AdS (RNAdS) black hole.

The spacetime is compact in θ direction, θ being an azimuthal angle (θ and $\theta + 2n\pi/k$, $n \in Z$, represent the same set of points in the spacetime). Therefore, the scalar fields are continuous periodic functions of the coordinate θ. The non-negativity of Φ^2 is also guaranteed. For simplicity, and with no loss o generality, we may choose $c_0 = 0$, $c_1 = 1$ and $c_2 = 0$. Moreover, it is possible to make $k = 1$ by a re-parameterization of the coordinates: $kt \mapsto t$, $r/k \mapsto r$, $k\theta \mapsto \theta$. The resulting metric is exactly the equatorial plane of the 4D Reissner-Nordström-AdS black hole, justifying the name used here. Both of the scalar fields are continuous periodic functions of the azimuthal angle θ. Hence, the length of a closed spacelike curve on which $t =$ constant and $r =$ constant is $2\pi r$, and the spacetime may have (circular) horizons depending on the relative values of the parameters m, q, and g, and has a polynomial singularity at $r = 0$ (see below).

It is worth mentioning once more that the second derivative of the dilaton field is a continuous function of θ. This fact is important for the continuity of the energy momentum tensor (28) which depends explicitly on $\ddot{\Phi}$ and $\dot{\Psi}$. This can also be seen from equations (23) and (25), from what we see that if Φ_2 is a continuous function of θ, it holds also for $\ddot{\Phi}_2$ and $\dot{\Psi}_2$, implying the continuity of the EMT (see e. g. [30]).

The present 3D RNAdS black hole has several interesting properties and we study them in some detail in the following. The spacetime metric is given by (40), and the physical interpretation of parameters m, q and g is given below. The parameter k^2 is kept in the equations, even though it can be made equal to unity.

6.1. Metric and charges:

In order to investigate the main properties of such a spacetime we start computing its global charges.

To define the mass for the metric (40) we note that the one-dimensional boundary $\mathcal{S}_1$ is defined by the hypersurface $t =$ constant, $r =$ constant, in the asymptotic limit $r \longrightarrow \infty$. The induced metric on $\mathcal{S}_1$, as embedded in D_2, σ_{ab} is obtained from (40) by putting $dt = 0$ and $dr = 0$. Thus, $a, b = 2$ and $\sigma_{ab} = \sigma_{22} \equiv \sigma = r^2$.

Comparing equations (40) to (9) and using equation (10) we get the extrinsic curvature of $\mathcal{S}_1$ modified by the dilaton $\kappa^\Phi = -2\sqrt{\Phi^2 F/r^2}$, where F is given by $F = \alpha^2 r^2 + k^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}$. Substituting κ^Φ into (7) and integrating over the infinite boundary $\mathcal{S}_1$ we get the mass of the 3D black hole

$$M = \frac{2L_3}{4\pi} m \int_0^{\frac{\pi}{k}} \cos k\theta \;\, d\theta = m\,, \tag{44}$$

where we have put $L_3 = \pi k$. By comparison, it is seen that this 3D mass is the same as the mass of the 4D Reissner-Nordström-AdS black hole.

Since metric (40) is static, it follows from (8) that the angular momentum is zero, $J = 0$.

The electric charge comes from equation (12) and is $Q_e = q$.

The full dilaton field Φ and the background dilaton field Φ_0 are equal $\Phi = \Phi_0 = r\cos k\theta$, implying that the dilaton charge is zero.

The charges for the scalar field Ψ are given by equations (13) and (14). In the case under consideration we obtain $Q_\Psi = 0$, and $\tilde{Q}_\Psi = g$.

The above metric has horizons at points where $F(r) = 0$ possibly indicating the presence of a black hole. This is, in fact, the 3D Reissner-Nordström-AdS black hole, i.e., a static charged black hole immersed in an asymptotically AdS spacetime

6.2. Singularities, horizons, and causal structure:

The causal structure of the spacetime given by metric (40) is very similar to the 4D Reissner-Nordström-anti de Sitter black hole. In order to see this, we show here the singularities and horizons of such a spacetime.

We start computing the Ricci and the Kretschmann scalars. The result is the same as in the section 4, given in equations (32) and (33). Thus, there is a singularity at $r = 0$ (R and K diverge at $r = 0$).

The solution has totally different character depending on whether $r > 0$ or $r < 0$. The important black hole solution exists for $r > 0$ which case we analyze now.

The horizons are given by the solutions of the equation

$$\Delta = \alpha^2 r^4 + k^2 r^2 - 2m\,r + q^2 + g^2 = 0\,. \tag{45}$$

By restricting the analysis to $r \geq 0$, the solutions of the above equation can be classified into three different cases:

(i) If $m^2\left[k^6 + 27\alpha^2 m^2 - 36k^2\alpha^2(q^2+g^2)\right] > (q^2+g^2)\left[k^4 - 4\alpha^2(q^2+g^2)\right]^2$, equation (45) has two real positive roots, r_+ and r_-, corresponding respectively to an event horizon and a Cauchy (inner) horizon. The spacetime is then a 3D spherical black hole whose geodesic and causal structures are the same as for the equatorial plane of the 4D Reissner-Nordström-anti de Sitter black hole. Singularity at $r = 0$ is hidden by the horizon.

(ii) In the extreme case, parameters α^2, m and q are related by the constraint $m^2\left[k^6 + 27\alpha^2 m^2 - 36k^2\alpha^2(q^2+g^2)\right] = (q^2+g^2)\left[k^4 - 4\alpha^2(q^2+g^2)\right]^2$. This is the black hole in which the two horizons coincide and are given by

$$r_+ = r_- = \frac{k^2 m + 12m\,\alpha^2(q^2+g^2)}{k^2 + 18\alpha^2 m^2 - 4k^2\alpha^2(q^2+g^2)}\,. \tag{46}$$

The singularity is hidden by the null horizon at $r = r_+ = r_-$, and the causal structure is the same as the equatorial plane of the 4D extreme Reissner-Nordström-AdS black hole.

(iii) For $m^2\left[k^6+27\alpha^2 m^2-36k^2\alpha^2(q^2+g^2)\right]<(q^2+g^2)\left[k^4-4\alpha^2(q^2+g^2)\right]^2$, the metric is well behaved over the whole range from the singularity $r=0$ to the asymptotic limit $r\longrightarrow\infty$. There are no horizons and the spacetime is a naked singularity.

From the above description, the Penrose diagrams, with the inherent topology and causal structure of spacetime, can easily be drawn.

6.3. The $\alpha=0$ cases: The 3D Reissner-Nordström and the Schwarzschild black holes

A simple but still interesting 3D spacetime that follows from the previous solutions is the 3D black hole obtained from (40) in the case $\alpha^2=0$,

$$ds^2=-(k^2-\frac{2m}{r}+\frac{q^2+g^2}{r^2})dt^2+\frac{dr^2}{k^2-\frac{2m}{r}+\frac{q^2+g^2}{r^2}}+r^2d\theta^2\,, \tag{47}$$

$$\tag{48}$$

with $\mathbf{A}$, Φ and Ψ given respectively by equations (41), (42) and (43). This result can also be seen as the hypersurface $\theta=\pi/2$ of the 4D Reissner-Nordström (RN) black hole. Its singularities, horizons and causal structure are the same as the equatorial plane of the 4D RN black hole.

For $q^2+g^2\le m^2$, metric (47) has two horizons at $r=r_\pm$, given by $|k|\,r_\pm=m\pm\sqrt{m^2-q^2-g^2}$, corresponding respectively to event and Cauchy horizons. As in the case $\alpha^2\ne 0$, there is a singularity at $r=0$ which is hidden (to external observers at $r\longrightarrow\infty$) by the event horizon at $r=r_+$.

Whenever the condition $q^2+g^2>m^2$ is satisfied, the spacetime is a naked singularity.

If $q^2+g^2=0$, the solution obtained from (47) is a 3D Schwarzschild spacetime, with the same causal structure, horizon and singularity as the 4D Schwarzschild black hole.

6.4. The uncharged case, $g^2+q^2=0$

If $q^2+g^2=0$, the solution obtained from (40) is a 3D Schwarzschild-AdS spacetime, with the same causal structure, horizon and singularity as the 4D Schwarzschild-AdS black hole (also known as Kottler spacetime)

$$ds^2=-\left(k^2-\frac{2m}{r}+\alpha^2r^2\right)dt^2+\frac{1}{k^2-\frac{2m}{r}+\alpha^2r^2}dr^2+r^2d\theta^2\,, \tag{49}$$

where Φ is given by (42), and the other fields being zero.

If we also have $m=0$, the solution is the (3D) anti-de Sitter metric with no other charges nor fields besides the dilaton and the cosmological constant.

7. Planar 3D black holes - II

The solution given by equation (40)–(43) admits a different kind of topology besides the spherical one considered in the preceding section.

In order to see that, define a new coordinate $x = Lk\theta$ with $-\infty \leq x < \infty$, L being an arbitrary constant carrying dimensions of length, and choose scalar fields as in equations (42) and (43):

$$\Phi = r\left(c_0 \sin(x/L) + c_1 \cos(x/L)\right) , \tag{50}$$

$$\Psi = gL\left(-c_0 \cos(x/L) + c_1 \sin(x/L)\right) + c_2 . \tag{51}$$

The metric and gauge fields are given by equations (40) and (41), respectively.

The resulting spacetime is then a planar black hole, analogous to the case reported in section 5.

The mass and charges can be defined only as linear densities along the infinite boundary (The line C_θ, at the asymptotic region $r \longrightarrow \infty$). For instance, the mass within a length L of the boundary is $M_L = \frac{2L_3}{4\pi} m \int_{\frac{-\pi L}{2}}^{\frac{\pi L}{2}} \cos\frac{x}{L}\, dx = mL$ (we took $L_3 = \pi$). As in section 5, the parameter m is then the mass per unit length, while q and g are the charge densities associated respectively to the gauge field $\mathbf{A}$ and to the scalar field Ψ.

The local geometric properties and the causal structure of this solution is, however, very similar to the 3D RNAdS black hole studied in section 6, and we do not discuss the details here.

8. Planar 3D black hole solutions - III

The general form of the solution obtained when we choose $\frac{\ddot{\Phi}_2}{\Phi_2} = +k^2 > 0$ is [see equations (101) in the Appendix]

$$ds^2 = \left(-k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(-k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\right)} + r^2 d\theta^2 , \tag{52}$$

$$\mathbf{A} = -\frac{q}{r}\mathbf{d}t , \tag{53}$$

$$\Phi = r\ (c_0 \sinh k\theta + c_1 \cosh k\theta) \tag{54}$$

$$\Psi = \frac{g}{k}\left(c_0 \cosh k\theta + c_1 \sinh k\theta\right) + c_2 , \tag{55}$$

where m, c_0, c_1, c_2, g, q and k are generic integration constants. The coordinates can be normalized such that $k^2 = 1$. The solution corresponds to the hypersurface $\rho =$ constant of the topological black hole with genus $\mathsf{g} > 1$ of reference [40] (see also [41]). The space surface $t =$ constant, $\rho =$ constant of such a 4D black hole has the topology of a hyperbolic two-space H^2. The functional form of Φ also justifies the adjective hyperbolic used in the present case.

The spacetime cannot be compactified in θ direction, because the dilaton field would be a multivalued function.

The choice $c_0 = 0$, $c_1 > 0$, and θ in the range $-\infty \leq \theta < +\infty$, assures the dilaton field is a continuous and non-zero function of the coordinates. Moreover, $\ddot{\Phi}$ and $\dot{\Psi}$ are continuous functions of θ, implying the continuity of the EMT, what guarantees the basic properties of a physical spacetime (see e. g. [30]).

As in the case of section 5, mass, and charges are infinite, but the linear densities of these quantities are finite and well defined.

The locus $r = 0$ is a true spacetime singularity, since the Ricci and Kretschmann scalars are the same as in section 5 (given respectively by equations (38) and (39)). Moreover, there are metric singularities (horizons) at points where

$$\Delta = \alpha^2 r^4 - r^2 - 2m\, r + q^2 + g^2 = 0\,. \tag{56}$$

Depending on the relative values of the parameters, α^2, m, q and g, there may be two horizons and the solution is an asymptotically anti-de Sitter black hole. Even though these very interesting properties, a more exhaustive analysis of such a black hole will not be presented in this paper.

9. Rotating charged 3D black holes

9.1. The rotating metric

The circular black holes discussed in sections 4 and 6 can be put to rotate. In order to add angular momentum to the spacetime we perform a local coordinate transformation which mixes time and angular coordinates as follows (see e.g. [10, 11, 42, 43])

$$\begin{aligned} & t \mapsto t - \frac{\omega}{\alpha^2}\theta\,, \\ & \theta \mapsto \gamma\theta\,, \end{aligned} \tag{57}$$

where γ and ω are constant parameters.

The next step is substituting the transformation (57) into the equations of each one of the above static solutions. For instance, taking (40)–(43) we find

$$ds^2 = -F\left(dt - \frac{\omega}{\alpha^2}d\theta\right)^2 + \gamma^2 r^2 d\theta^2 + \frac{dr^2}{F}\,, \tag{58}$$

$$\mathbf{A} = -\frac{q}{r}\left(\mathbf{d}t - \frac{\omega}{\alpha^2}\mathbf{d}\theta\right)\,, \tag{59}$$

$$\Phi = r\cos\left(\gamma\, k\theta\right)\,, \tag{60}$$

$$\Psi = -\frac{g}{k}\sin\left(\gamma\, k\theta\right)\,, \tag{61}$$

where

$$F = k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\,, \tag{62}$$

and we chose $c_0 = 0$, $c_1 = 1$ and $c_2 = 0$.

Even though we used the solution (40)–(43) explicitly, the constant k^2 is kept explicitly in the metric, so that the two particular cases $k^2 = 0$ (section 4), and $k^2 > 0$ (section 6) are both contained in the analysis. The case $k = 0$ is obtained by taking the appropriated limit in the above equations. In that case, one also has to put $g = 0$, i.e., $\Psi = 0$, and $\Phi = r$ for all $\theta \in (0, 2\pi)$, as shown in section 4.

Planar black holes such as those reported in sections 5, 7 and 8 cannot be put to rotate.

Let us mention that parameters γ and ω are not independent and one of them can be redefined by a re-parameterization of the coordinates. An interesting choice for $k^2 \neq 0$ is $\gamma^2(1-k^2) = \frac{\omega^2}{\alpha^2}$, what ensures the asymptotic form of the metric for large r is such that the length of a closed curve $t =$ constant, $r =$ constant, is exactly $2\pi r$. Another possible and simple choice is to take $\gamma = 1$, leaving ω free. This last choice is used throughout this paper because it applies also to the case $k^2 = 0$.

Analyzing the Einstein-Rosen bridge of the static solution one concludes that the spacetime is not simply connected which implies that the first Betti number of the manifold is one, i.e., closed curves encircling the horizon cannot be shrunk to a point. Therefore, transformations (57) generate a new metric because they are not permitted global coordinate transformations [44]. Metrics (40) and (58) are distinct, for they can be locally mapped into each other but not globally.

9.2. The global charges

In the spacetime associated to the metric (58) we choose a region $\mathcal{M}$ of spacetime bounded by $r =$ constant, and two space-like surfaces $t = t_1$ and $t = t_2$. The region of the spacetime $t =$ constant, $r =$ constant, is the one-dimensional boundary $\mathcal{S}_1$ of the two-space D_2. The boundary of $\mathcal{M}$, $\partial\mathcal{M}$, in the present case consists of the product of $\mathcal{S}_1$ with timelike lines ($r =$ constant$\,, \theta =$ constant) joining the surfaces $t = t_1$ and $t = t_2$, and these two surfaces themselves. The metric σ_{ab} is obtained from (58) by making $dt = 0$ and $dr = 0$, while the two-space D_2 metric h_{ij} $(i, j = 1, 2)$ is obtained by putting $dt = 0$. The timelike and spacelike unit vectors t^i and n^i are, respectively, $t^i = \frac{\delta^i_0}{N} - \frac{N^\theta \delta^i_2}{N}$, $n^i = \sqrt{F}\delta^i_1$. Metric (58) admits the two Killing vectors, a spacelike η_θ and a timelike η_t, needed in order to define mass and angular momentum.

Comparing metric (58) to (9) and using (10) we get the following expression for the extrinsic curvature of $\mathcal{S}_1$,

$$\kappa^\Phi = -\frac{1}{2}\sqrt{F\Phi^2}\left[\frac{2r\left(1-\frac{\omega^2}{\alpha^2}\right) - \frac{\omega^2}{\alpha^4}\left(\frac{2m}{r^2} - 2\frac{q^2+g^2}{r^3}\right)}{r^2\left(1-\frac{\omega^2}{\alpha^2}\right) - \frac{\omega^2}{\alpha^4}\left(k^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\right)} + \frac{2}{r}\right], \tag{63}$$

where F and Φ are given respectively by (62) and (60). Let us mention that the above equation holds for both of the classes of spacetimes presented in sections 4 and 6.

To build $\delta\left(\kappa^\Phi\right) = \kappa^\Phi - \left(\kappa^\Phi\right)_o$ we take κ^Φ from equation (63), which was obtained from the full solution given in equations (58)–(61). The extrinsic curvature of the background spacetime, $\left(\kappa^\Phi\right)_o$, is obtained from the 3D spacetime with no black hole present, and follows from the same relation (63) by choosing $m = 0$, $q = 0$ and $g = 0$. Then, substituting $\delta\left(\kappa^\Phi\right)$ into (7), and taking the limit $r \longrightarrow \infty$, the mass of the toroidal 3D black hole is finally obtained,

$$M = m\left(1 + \frac{3}{2\alpha^2}\frac{\omega^2}{1-\frac{\omega^2}{\alpha^2}}\right), \tag{64}$$

where m is the mass of the static black hole, and to simplify we have put $L_3 = \pi|k|$ for $k^2 > 0$, and $L_3 = 2$ in the case $k^2 = 0$. Assuming $0 \leq \frac{\omega^2}{\alpha^2} < 1$, we see that the rotating

black hole mass is larger than the mass of the original static black hole. The additional mass, $\delta M = \frac{3m}{2\alpha^2}\frac{\omega^2}{1-\frac{\omega^2}{\alpha^2}}$, depends explicitly on the rotation parameter ω and can be viewed as being generated by the motion of the 3D system.

To calculate the angular momentum we compare metric (58) to (9) and use equation (11) which gives

$$j_i^{\Phi}\eta_\theta^i = \frac{\omega}{\alpha^2}\sqrt{\Phi^2}\frac{(F'r-2F)}{\sqrt{r^2-\frac{\omega^2 F}{\alpha^2}}}\,, \tag{65}$$

where F' indicates the derivative with respect to r. From the above result and equation (8) the angular momentum follows

$$J = \frac{3}{2}m\frac{\omega}{\alpha^2}\,. \tag{66}$$

However, it should be emphasized that the angular momentum as defined by equation (11) is shown to be a conserved quantity only in the case $k=0$, when the dilaton field is rotationally invariant [33]. The dilaton field must be constant on orbits of the Killing vector $\eta_\theta = \frac{\partial}{\partial\theta}$, which excludes the case corresponding to $k^2 \neq 0$.

Other nonzero charges are the same as for the static black hole, $Q_e = q$, and $\tilde{Q}_\Psi = g$.

9.3. Causal Structure of the Charged Rotating Spacetime

We can see that the metric for the charged rotating spherically symmetric 3D asymptotically anti-de Sitter spacetime, given in (58), has the same singularities and horizons as the metric of the non-rotating spacetime given by (40). Of course, there are differences and we will explore them here.

The coordinate transformation (57) does not change local properties of the spacetime. Hence, the Kretschmann scalar K is the same as for the non-rotating spacetime, and so are the singularities.

The horizons of the rotating metric (58) are located at points where $F(r)=0$. Hence, the horizons are still given by the solutions of equation (45), the same as for the static spacetime.

The infinite redshift surfaces, which are given by the zeros of the metric coefficient g_{tt}, are also the same for both the rotating and the static spacetime. Such a surface corresponds to each one of the roots of the equation $F(r)=0$, and coincides with the horizons.

The causal structure of static and rotating spacetimes are very similar indeed. The differences appear when one studies the existence of closed timelike curves (CTCs).

To study closed timelike curves (CTCs) we first note that the angular Killing vector ∂_θ has norm given by $\partial_\theta.\partial_\theta = g_{\theta\theta} = r^2 - \omega^2 F/\alpha^4$. There are CTCs for $g_{\theta\theta}<0$. The radii for which $g_{\theta\theta}=0$ are given by the solutions of the equation

$$\alpha^2 r^4\left(1-\frac{\omega^2}{\alpha^2}\right) - \frac{\omega^2}{\alpha^2}k^2\,r^2 + 2\frac{\omega^2}{\alpha^2}m\,r - \frac{\omega^2}{\alpha^2}\left(q^2+g^2\right) = 0\,. \tag{67}$$

It is a quartic equation and one can easily find its zeros in terms of the other parameters. To avoid writing a new set of equations for these zeros, we just comment on some interesting cases one may find.

To begin with, we consider firstly the case $k^2 > 0$ and choose the other parameters in such a way that the spacetime has two horizons, r_- and r_+. Since we fixed $0 \leq \frac{\omega^2}{\alpha^2} < 1$, the solutions of last equation are threefold depending on the discriminant

$$D = m^2 \left\{ \frac{\omega^2}{\alpha^2} k^4 \left(3 - \frac{\omega^2}{\alpha^2} k^6 \right) + 18\alpha^2 \left(1 - \frac{\omega^2}{\alpha^2} \right) \left[2 \left(q^2 + g^2 \right) \left(1 + \frac{\omega^2}{\alpha^2} k^6 \right) - 3m^2 \frac{\omega^2}{\alpha^2} k^4 \right] \right\} \\ - 2 \left(q^2 + g^2 \right) \left[1 + 4\alpha^2 k^2 \left(q^2 + g^2 \right) \left(1 - \frac{\omega^2}{\alpha^2} \right) \right]^2 . \tag{68}$$

The three possibilities for the real roots of (67) are: (i) If $D > 0$ there are three positive roots; (ii) for $D = 0$ two positive roots; and (iii) one positive root if $D > 0$. We assume $\alpha^2 > 0$ and $m > 0$.

Firstly, if $D < 0$, equation (67) has just one real positive zero, r_0, say. This happens in particular when ω^2/α^2 is small compared to unity, and for a large range of the other parameters. In this case, r_0 is smaller than the horizon r_+ for all the range of the other parameters. The CTCs are located in the region $r < r_0$, so there are CTCs just inside the event horizon. Moreover, there are CTCs outside the inner horizon, $r_0 > r_-$, just when the charge is very small compared to the mass, otherwise the CTCs happen just inside the inner horizon.

In the case $D = 0$, equation (67) has two positive roots. Let us name them r_0 and r_1 and assume $r_1 > r_0$. Similarly to the case $D < 0$ above, there are CTCs in the region $r < r_0$. However, in the region which the radial coordinate r assumes values between r_0 and r_1, $r_0 < r < r_1$, there are two possibilities depending on the values of the discriminant

$$D_1 = 6k^2\alpha^2 \left(q^2 + g^2 \right) \left(1 - \frac{\omega^2}{\alpha^2} \right) - \left[\frac{\omega^2}{3\alpha^2} k^4 + 4\alpha^2 \left(1 - \frac{\omega^2}{\alpha^2} \right) m \right]^2 .$$

For $D_1 > 0$ the CTCs are restricted to the region $r < r_0$ and r_0 is inside the horizon r_+. If $D_1 < 0$ there are CTCs in all the region $0 \geq r < r_1$, and r_1 may be larger than the horizon.

The third case, $D > 0$ is when equation (67) has three positive solutions, r_0, r_1 and r_2. There are CTCs in both of the regions $r_1 < r < r_2$ and $r < r_0$. In the region $r_0 < r < r_1$ there are no CTCs. The values of r_1 and r_2 depend strongly on the parameter ω^2, and for ω^2/α^2 sufficiently close to unity they can be both larger than the event horizon.

Now, for $k^2 = 0$, equation (67) has just one positive root r_0, say, which depends strongly on the values of the charges q and g, and vanishes when $g^2 + q^2 = 0$. Moreover, the CTCs are restricted to the region inside the event horizon. For $r > r_0$ there are no CTCs.

10. Conclusions

Using a 3D dilaton-gravity Lagrangian, with additional scalar and gauge fields, inspired in the dimensional reduction of the 4D Einstein-Maxwell Lagrangian we obtained new 3D static black hole solutions.

Assuming an ansatz where the spacetime is static and has circular symmetry, but allowing the other fields to be anisotropic, we show that the field equations do not impose the circular symmetry on the fields. In particular, the dilaton field needs not be isotropic. By using spherical coordinates (r, θ), the anisotropy of the static dilaton field shows up through its explicit dependence on the angular coordinate. There are five different forms of the dilaton as a function of r and θ, which are compatible with the symmetry of the spacetime. Namely, a constant; a function of r only; a function of r times a linear function of theta; a function of r times a circular function of θ; and a function of r times a hyperbolic function of θ.

As a consequence, we find at least six different families of black holes, each one characterized by a different dilaton field.

The first is the well known BTZ family of black holes, which follows when we choose the dilaton to be constant.

Secondly, when the dilaton is isotropic (depends only on the radial coordinate), it is found a 3D spherical black hole solution which corresponds to the 4D cylindrical (or toroidal) black hole studied in [1] with one dimension suppressed.

The dilaton can also be a linear function on of θ, which is not a compact coordinate. The solution may be interpreted as a planar asymptotically AdS black hole.

Another very interesting black hole solution is found when we choose the dilaton dependence upon θ to be the cosine function. This implies in a 3D spherically symmetric spacetime which corresponds to the equatorial plane of the 4D Reissner-Nordström-AdS black hole. Changing the topology of the spacetime in the θ direction to be the real line, the fifth class of solutions is found as a variant to the 3D RNAdS case.

The last class of 3D black hole spacetimes, which has planar topology, is obtained in the case the dilaton is a hyperbolic function of the angular coordinate θ. Such a spacetime corresponds to the 4D charged hyperbolic black hole of references [40, 41, 45] with one dimension suppressed.

The spherical black holes can be put to rotate by a coordinate transformation which mixes time and angle, but only in cases when the dilaton is isotropic the angular momentum is a conserved quantity.

Acknowledgments

One of us (V.T.Z.) thanks Centro Multidisciplinar de Astrofísica at Instituto Superior Técnico (CENTRA-IST) for a grant and for hospitality while part of this work has been done. A.S.M. thanks the Brazilian Agency CAPES for a grant. We are indebted to J. P. S. Lemos for many interesting discussions and encouragement.

Appendix: General equations and solutions

The 3D metric is initially assumed to be static but without rotational symmetry, which in Schwarzschild-like coordinates reads

$$ds^2 = -f^2\,dt^2 + g^2\,dr^2 + h^2\,d\theta^2\,, \tag{69}$$

where f, g, and h are functions of the coordinates r and θ only. The gauge field $\mathbf{A}$ is chosen in the form $\mathbf{A} = A_t(r,\theta)\mathbf{d}t$.

The Lagrangian $\mathcal{L}$ defined from equation (2) is then

$$\mathcal{L} = \frac{L_3}{16\pi}\Phi f g h\left[\frac{2}{h^2}\left(\frac{\dot f}{f}\frac{\dot h}{h} - \frac{\dot f}{f}\frac{\dot g}{g} + \frac{\dot g}{g}\frac{\dot h}{h} - \frac{\ddot f}{f} - \frac{\ddot g}{g}\right) + \frac{2}{g^2}\left(\frac{f'}{f}\frac{g'}{g} + \frac{g'}{g}\frac{h'}{h} - \frac{\dot f}{f}\frac{\dot h}{h} - \frac{f''}{f} - \frac{h''}{h}\right)\right.$$
$$\left. +6\alpha^2 + \frac{2}{f^2}\left(\frac{A_t'^2}{g^2} + \frac{\dot A_t^2}{h^2}\right) - \frac{2}{\Phi^2}\left(\frac{\Psi'^2}{g^2} + \frac{\dot\Psi^2}{h^2}\right)\right]\,, \tag{70}$$

where a prime denotes partial derivative with respect to r, and a dot denotes partial derivative with respect to θ.

Now we calculate the equations of motion (EOM) as defined in section 2.

From the metric (69) we get the following non-zero components of the Einstein tensor

$$G_t^t = \frac{1}{h^2}\frac{\ddot g}{g} + \frac{1}{g^2}\frac{h''}{h} - \frac{1}{g^2}\frac{g'}{g}\frac{h'}{h} - \frac{1}{h^2}\frac{\dot g}{g}\frac{\dot h}{h}\,, \tag{71}$$

$$G_r^r \quad = \frac{1}{h^2}\frac{\ddot f}{f} - \frac{\dot f}{f}\frac{\dot h}{h^3} + \frac{1}{g^2}\frac{f'}{f}\frac{h'}{h}\,, \tag{72}$$

$$G_\theta^r = G_r^\theta\frac{h^2}{g^2} = -\frac{1}{g^2}\frac{\dot f'}{f} + \frac{1}{g^2}\frac{\dot g}{g}\frac{f'}{f} + \frac{1}{g^2}\frac{\dot f}{f}\frac{g'}{g}\,, \tag{73}$$

$$G_\theta^\theta = \frac{1}{g^2}\frac{f''}{f} - \frac{1}{g^2}\frac{f'}{f}\frac{g'}{g} + \frac{1}{h^2}\frac{\dot f}{f}\frac{\dot g}{g}\,, \tag{74}$$

Before writing explicitly the EMT components and the EOM for the fields, we restrict the metric to have spherical symmetry we have $f = 1/g = F(r)$, $h = r$. In such a case, the nonzero components of the Einstein tensor are $G_t^t = G_r^r = F'/2r$, $G_\theta^\theta = F''/2$.

Moreover, to simplify the analysis, the other fields are split into the product of two independent functions of each one of the coordinates r and θ, $A_t = A_1(r)\,A_2(\theta)$, $\Phi = \Phi_1(r)\,\Phi_2(\theta)$, $\Psi = \Psi_1(r)\,\Psi_2(\theta)$. Substituting this ansatz into the Einstein equations (3) we get

$$G_t^t = \frac{F'}{2r} = 3\alpha^2 - {A_1}'^2{A_2}^2 - \frac{{\dot A_2}^2{A_1}^2}{r^2F} - F\frac{{\Phi_1}''}{\Phi_1} - \frac{F'}{2}\frac{{\Phi_1}'}{\Phi_1} - \frac{F}{r}\frac{{\Phi_1}'}{\Phi_1} - \frac{1}{r^2}\frac{\ddot\Phi_2}{\Phi_2}$$
$$- F\frac{{\Psi_1}'^2\Psi_2^2}{{\Phi_1}^2{\Phi_2}^2} - \frac{1}{r^2}\frac{{\Psi_1}^2\dot\Psi_2^2}{{\Phi_1}^2{\Phi_2}^2}\,, \tag{75}$$

$$G_r^r = \frac{F'}{2r} = 3\alpha^2 - {A_1}'^2{A_2}^2 + \frac{{\dot A_2}^2{A_1}^2}{r^2F} - \frac{F'}{2}\frac{\Phi_1'}{\Phi_1} - \frac{F}{r}\frac{{\Phi_1}'}{\Phi_1} - \frac{1}{r^2}\frac{\ddot\Phi_2}{\Phi_2}$$
$$+ F\frac{{\Psi_1}'^2{\Psi_2}^2}{{\Phi_1}^2{\Phi_2}^2} - \frac{1}{r^2}\frac{{\Psi_1}^2\dot\Psi_2^2}{{\Phi_1}^2{\Phi_2}^2}\,, \tag{76}$$

$$G^r_\theta = 0 = -2A_1A_2A_1'\dot{A}_2 + F\frac{\Phi_1'}{\Phi_1}\frac{\dot{\Phi}_2}{\Phi_2} - \frac{F}{r}\frac{\dot{\Phi}_2}{\Phi_2} + 2F\frac{\Psi_1\Psi_2}{\Phi_1^2\Phi_2^2}\Psi_1'\dot{\Psi}_2\,, \tag{77}$$

$$G^\theta_\theta = \frac{F''}{2} = 3\alpha^2 - {A_1}'^2{A_2}^2 + \frac{\dot{A_2}^2{A_1}^2}{r^2F} - F\frac{{\Phi_1}''}{\Phi_1} - F'\frac{{\Phi_1}'}{\Phi_1} - F\frac{{\Psi_1}'^2\Psi_2^2}{{\Phi_1}^2{\Phi_2}^2} + \frac{1}{r^2}\frac{{\Psi_1}^2\dot{\Psi}_2^2}{{\Phi_1}^2{\Phi_2}^2}\,, \tag{78}$$

where now the prime and the dot denote total derivative with respect to r and θ, respectively.

Variation of the action $S = \int \mathcal{L}\, d^3x$ with respect to Φ gives the constraint

$$R + 6\alpha^2 - \mathbf{F}^2 + 2\Phi^{-2}(\nabla\Psi)^2 = 0\,, \tag{79}$$

which in the present case assumes the form

$$\frac{2F'}{r} + F'' - 6\alpha^2 - {A_1'}^2{A_2}^2 - \frac{{A_1}^2\dot{A}_2^2}{r^2F} - \frac{2}{{\Phi_1}^2{\Phi_2}^2}\left({\Psi_1'}^2{\Psi_2}^2 + {\Psi_1}^2\dot{\Psi}_2^2\right) = 0\,. \tag{80}$$

The above constraint is in fact one of the equations of motion. Variation of the action with respect to g_{ij}, A_t and Ψ completes the set of EOM, given respectively by equations (3), (4) and (6). Equation (5) is obtained by substituting (3) into equation (79). The system of equations of motion as written in section 2 does not admit solutions without dilaton. In such a case, equation (79) is not part of the system of EOM, and neither is equation (5), so that the case $\Phi =$ constant has to be considered separately (see below).

The sole Maxwell equation for the gauge field $\mathbf{A}$, equation (4), is

$$A_2\left(A_1'' + A_1'\frac{1}{r} + A_1'\frac{\Phi_1'}{\Phi_1}\right) + \frac{A_1}{r^2F}\left(\ddot{A}_2 + \dot{A}_2\frac{\dot{\Phi}_2}{\Phi_2}\right) = 0\,. \tag{81}$$

The equations for the scalar fields Φ and Ψ, (5) and (6), respectively, reduce to

$$\begin{aligned} &-3\alpha^2 + F\frac{{\Phi_1}''}{\Phi_1} + \frac{{\Phi_1}'}{\Phi_1}\left(\frac{F'}{2} + \frac{F}{r}\right) - \frac{1}{r^2}\frac{\ddot{\Phi}_2}{\Phi_2} - F\frac{{\Psi_1}'^2\Psi_2^2}{{\Phi_1}^2{\Phi_2}^2} - \frac{1}{r^2}\frac{\dot{\Psi_1}^2{\Psi_2}^2}{{\Phi_1}^2{\Phi_2}^2} \\ &-{A_1}'^2{A_2}^2 - \frac{\dot{A_2}^2{A_1}^2}{r^2F} = 0\,, \end{aligned} \tag{82}$$

$$\frac{{\Psi_1}''}{\Psi_1} + \frac{F'}{F}\frac{{\Psi_1}'}{\Psi_1} + \frac{1}{r}\frac{{\Psi_1}'}{\Psi_1} - \frac{\Phi_1'}{\Phi_1}\frac{\Psi_1'}{\Psi_1} + \frac{1}{Fr^2}\left(\frac{\ddot{\Psi}_2}{\Psi_2} - \frac{\dot{\Psi}_2}{\Psi_2}\frac{\dot{\Phi}_2}{\Phi_2}\right) = 0\,. \tag{83}$$

Equations (75)–(83) are the basic equations to be solved. We shall see, however, that this system of equations can be significantly simplified after a careful analysis.

From equations (82) and (75) we find

$$F'\left(\frac{1}{r} - \frac{\Phi_1'}{\Phi_1}\right) = 0\,. \tag{84}$$

The solution of this equation is twofold: (i) $F(r) =$ constant and $\Phi_1(r)$ arbitrary, or (ii) $F(r)$ arbitrary and $\Phi_1(r) = c_0 r$, where c_0 is a nonzero constant. The case (i) is not interesting, because it leads to a flat spacetime. Hence, we take the second possibility $\Phi_1(r) = c_0 r$ and leave $F(r)$ to be determined by the set of remaining equations.

Equations (75) and (76) yield

$$F\frac{\Phi_1''}{\Phi_1} + A_1^2\frac{\dot{A}_2^2}{r^2F^2} + \Psi_2^2\frac{{\Psi_1'}^2}{\Phi_1^2\Phi_2^2} = 0\,, \tag{85}$$

which, together with equation (84), implies $A_2 =$ constant and $\Psi_1 =$ constant. These constants can be made equal to unity, since they are absorbed by the functions A_1 and Ψ_2, respectively, in the definitions $A_t(r,\theta) = A_1(r)A_2(\theta)$ and $\Psi(r,\theta) = \Psi_1(r)\Psi_2(\theta)$. Hence, the gauge potential A_t depends only on the coordinate r, while Ψ depends only on the coordinate θ.

By rewriting the field equations including these results, the system of seven field equation (75)–(83) may then be replaced by the following equivalent set

$$\frac{F'}{2}\left(\frac{1}{r}+\frac{\Phi_1'}{\Phi_1}\right) - 3\alpha^2 + {A_1'}^2 + \frac{F}{r}\frac{\Phi_1'}{\Phi_1} + \frac{1}{r^2}\frac{\ddot{\Phi}_2}{\Phi_2} + \frac{{\Psi_1}^2}{r^2}\frac{{\dot{\Psi}_2}^2}{{\Phi_1}^2{\Phi_2}^2} = 0\,, \tag{86}$$

$$\frac{1}{2}F'' + \frac{F'}{2}\left(\frac{1}{r}+\frac{3\Phi_1'}{\Phi_1}\right) - 6\alpha^2 - \frac{F}{r}\frac{\Phi_1'}{\Phi_1} + \frac{1}{r^2}\frac{\ddot{\Phi}_2}{\Phi_2} = 0\,, \tag{87}$$

$$A_1'\frac{\Phi_1'}{\Phi_1} + A_1'' + \frac{1}{r}A_1' = 0\,, \tag{88}$$

$$\dot{A}_2 = 0\,, \tag{89}$$

$$\Phi_1'' = 0\,, \tag{90}$$

$$\frac{\dot{\Phi}_2}{\Phi_2}\left(\frac{\Phi_1'}{\Phi_1} - \frac{1}{r}\right) = 0, \tag{91}$$

$$\Psi_1' = 0\,, \tag{92}$$

$$\ddot{\Psi}_2 - \dot{\Psi}_2\frac{\dot{\Phi}_2}{\Phi_2} = 0\,. \tag{93}$$

Even though there are eight equations for seven unknowns, there is no inconsistency, since the two first equations of the last system are not independent (see below).

The first two equations follow from the Einstein equations (75) and (78) after using the results of (84) and (85). Equation (88) follows by substituting $\dot{A}_2 = 0$ into Maxwell equation (81). Equation (91) is obtained from Einstein equation (77) after using the results implied by (85).

We observe that equations (90) and (91) are equivalent to the EOM for the dilaton, and includes the case $\Phi =$ constant as a particular solution. Notice also that (87), (89), (90), (91) and (92) imply the Einstein equation (76). The last two equations are equivalent to the EOM for Ψ (see equation (83)).

Function $\Phi_1(r)$ gives the dependence of the dilaton upon the coordinate r, which according to equations (90) and (91) is twofold: (i) $\Phi_1(r) =$ constant, in which case Φ_2 must also be a constant; (ii) $\Phi_1(r) = r$, in this case $\Phi_2(\theta)$ is not fixed by equation (91).

Now, equation (87) requires that

$$\frac{\ddot{\Phi}_2}{\Phi_2} = \pm k^2\,, \tag{94}$$

which together with (86) gives

$$\frac{\dot{\Psi}_2}{\Phi_2} = g\,, \tag{95}$$

k and g being arbitrary constants. The last equation is consistent with (93) and, in fact, replaces it.

The general solution for Φ in the case of spherical symmetry we are considering here can be split in four different particular cases, corresponding to the four different possibilities that follow from equations (90) and (94). Namely, (i) $\Phi_1' = 0$ and $\ddot{\Phi}_2 = 0$, (ii) $r\Phi_1' = \Phi_1$ and $\ddot{\Phi}_2 = 0$, (iii) $r\Phi_1' = \Phi_1$ and $\ddot{\Phi}_2 = +k^2 > 0$, and (iv) $r\Phi_1' = \Phi_1$ and $\ddot{\Phi}_2 = -k^2 < 0$.

Taking the results for the fields $\mathbf{A}$, Φ and Ψ into account we are left with the following equations for F

$$\frac{F'}{2}\left(\frac{1}{r}+\frac{\Phi_1'}{\Phi_1}\right) - 3\alpha^2 + \frac{q^2}{r^2\Phi_1^2} + \frac{F}{r}\frac{\Phi_1'}{\Phi_1} + \frac{k^2}{r^2} + \frac{g^2}{r^2{\Phi_1}^2} = 0\,, \tag{96}$$

$$\frac{1}{2}F'' + \frac{F'}{2}\left(\frac{1}{r}+\frac{3\Phi_1'}{\Phi_1}\right) - 6\alpha^2 - \frac{F}{r}\frac{\Phi_1'}{\Phi_1} + \frac{k^2}{r^2} = 0\,, \tag{97}$$

It is worth noticing that if $\Phi_1' = 0$ then equation (90) implies $\dot{\Phi}_2 = 0$ which requires $k^2 = 0$.

It is straightforward to show that the two above equations are functionally dependent. For, multiply (96) by r^2, differentiate with respect to r and use the fact that $\frac{r\Phi_1'}{\Phi_1} = \text{constant}$ to find

$$\frac{F''}{2}\left(1+\frac{r\Phi_1'}{\Phi_1}\right) + \frac{F'}{2}\left(\frac{1}{r}+\frac{3\Phi_1'}{\Phi_1}\right) - 6\alpha^2\left(1+\frac{r\Phi_1'}{\Phi_1}\right) + 2\frac{r\Phi_1'}{\Phi_1}\left(\frac{k^2}{r^2}+\frac{F'}{2}\frac{\Phi_1'}{\Phi_1}+\frac{F}{r}\frac{\Phi_1'}{\Phi_1}\right) = 0\,.$$

This last equation is identical to (97) in the two possible cases, $\frac{r\Phi_1'}{\Phi_1} = 1$ and $\frac{r\Phi_1'}{\Phi_1} = 0$. The last condition requires $k = 0$ in equation (94).

Now we write down the explicit solutions for each field considering the four different cases for dilaton field function mentioned above.

First consider $\Phi = \text{constant}$. The final solution in this case is

$$\begin{aligned}
F &= 3\alpha^2 r^2 + 2\left(q^2+g^2\right)\ln r - M\,,\\
\mathbf{A} &= q\ln r\;\mathbf{d}t\,,\\
\Phi &= c_0\,,\\
\Psi &= g\left(c_0\theta + c_1\right)\,.
\end{aligned} \tag{98}$$

The second class of solutions is that related to the choice $\Phi = r(c_0\theta + c_1)$. The relevant functions are

$$\begin{aligned}
F &= \alpha^2 r^2 - \frac{2M}{r} + \frac{q^2+g^2}{r^2}\,,\\
\mathbf{A} &= \frac{2q}{r}\;\mathbf{d}t\,,\\
\Phi &= r(c_0\theta + c_1)\,,\\
\Psi &= g\left(\frac{c_0}{2}\theta^2 + c_1\theta\right) + c_2\,.
\end{aligned} \tag{99}$$

The third type of solutions are obtained in the case $\ddot{\Phi}_2 < 0$ and is given by

$$F = k^2 + \alpha^2 r^2 - \frac{2m}{r} + \frac{q^2+g^2}{r^2}\,,$$

$$
\begin{aligned}
\mathbf{A} &= \frac{2q}{r}\,\mathbf{d}t\,,\\
\Phi &= r\left(c_0 \sin\, k\theta + c_1 \cos\, k\theta\right)\,,\\
\Psi &= -\,g\left(\frac{c_0}{k}\cos\, k\theta - \frac{c_1}{k}\sin\, k\theta\right) + c_2\,.
\end{aligned}
\tag{100}
$$

Finally, the fourth class of solutions follow in the case $\ddot{\Phi}_2 > 0$ and is of the form

$$
\begin{aligned}
F &= -\,k^2 + \alpha^2 r^2 - \frac{2M}{r} + \frac{q^2+g^2}{r^2}\,,\\
\mathbf{A} &= \frac{2q}{r}\,\mathbf{d}t\,,\\
\Phi &= r\left(c_0 \sinh\, k\theta + c_0 \cosh\, k\theta\right)\,,\\
\Psi &= g\left(\frac{c_0}{k}\cosh\, k\theta + \frac{c_1}{k}\sinh\, k\theta\right) + c_2\,.
\end{aligned}
\tag{101}
$$

The solution given by equations (98) is a simple generalization of the BTZ black hole. The second black hole solution, given in (99), was found in reference [1]. The two last cases, (100) and (101), are new solutions which may represent 3D black holes. In particular the case (100) is similar to the 4D Reissner-Nordström-AdS black hole.

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